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ON THE GENUS OF NILPOTENT GROUPS AND SPACES

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ABSTRACT
The notion of *genus*, applied to finitely generated nilpotent groups or to nilpotent spaces of finite type, was introduced by Mislin; he and the author showed how to introduce the structure of a finite abelian group into the genus if the group N has finite commutator subgroup. An example is given of a complete genus $N_0, N_1, \ldots, N_{s-1}$, which constitute a cyclic group generated by N_1, with the additional property that each N_i embeds in its successor as a normal subgroup with quotient cyclic of order l; of course, N_{s-1} embeds in N_0. The construction leads to the description of a family of nilpotent spaces $X_0, X_1, \ldots, X_{s-1}$, all in the same genus, no two of the same homotopy type, such that each X_i covers its successor as a cyclic l-sheeted regular covering; of course, X_{s-1} covers X_0. Here p is a prime, $n \geqq 1$, and $s = p^{n-1}(p-1)/2$, while l is semiprimitive module p^n.

0. Introduction

The notion of *genus*, applied to finitely generated nilpotent groups or to nilpotent spaces of finite type, was introduced by Mislin (see [4, 3]). Thus two finitely generated nilpotent groups N and M belong to the same genus if and only if, for each prime p, the localizations N_p and M_p are isomorphic; and a similar definition holds for nilpotent spaces of finite type. Strictly speaking, the genus should be regarded as consisting of isomorphism classes of groups or homotopy types of spaces. The detailed definitions may be found in [3].

In [4], Mislin showed how to calculate the order of the genus set of a nilpotent group N in the case that N has finite commutator subgroup, and this work was further developed in [2], where it was shown that the genus set, which is finite, admits a natural abelian group structure with the isomorphism class of N as

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neutral element. It follows from these calculations that the genus group is cyclic if the torsion subgroup of N is a p-group. In Section 1 of this paper we construct the entire genus of a certain nilpotent group N, given by

$$N = \langle x, y ; x^{p^{n+k}} = 1, yxy^{-1} = x^{u} \rangle. \tag{0.1}$$

Here p is a prime; n, $k \geqq 1$; $u = 1 + cp^k$, with $p \nmid c$; and we must exclude the exceptional case $p = 2$, $k = 1$. The genus set contains 1 element if $p = 2$, $n = 1$; otherwise it contains

$$s = \frac{p^{n-1}(p-1)}{2}$$

elements, and thus is non-trivial provided we exclude $p = 2$, $n = 1$; $p = 2$, $n = 2$; $p = 3$, $n = 1$.

We realize the entire genus as $N_0 (= N), N_1, \dots, N_{s-1}$; and, in the genus group,

$$N_i = iN_1.$$

However we do more, for we can construct an "Escher staircase" of normal embeddings

$$N_0 \xrightarrow{\phi_0} N_1 \to \cdots \to N_i \xrightarrow{\phi_i} N_{i+1} \to \cdots \to N_{s-1} \xrightarrow{\phi_{s-1}} N_0 \tag{0.2}$$

such that each quotient group is cyclic of order l, where l is *semi-primitive* modulo p^n (that is, the smallest power q of l such that $l^q \equiv \pm 1 \bmod p^n$ is $q = s$). Precisely,

$$N_i = \langle x, y ; x^{p^{n+k}} = 1, yxy^{-1} = x^{u^{m^i}} \rangle,$$

where $lm \equiv 1 \bmod p^n$, and $\phi_i : N_i \to N_{i+1}$ is given by $\phi_i x = x$, $\phi_i y = y^l$.

In Section 2 we first realize our Escher staircase by a sequence $M_0, M_1, \dots, M_{s-1}$ of $\mathbf{Z}C$-modules, where C is a cyclic infinite group, say $C = \langle \xi \rangle$. As abelian groups each of the M_i is $\mathbf{Z}/p^{n+k} = \langle a \rangle$, but the module structure in M_i is given by

$$\xi a = u^{m^i} a. \tag{0.3}$$

Following Cassidy [1] we may apply the notion of genus to a $\mathbf{Z}C$-module. Then the modules $M_0, M_1, \dots, M_{s-1}$ are pairwise non-isomorphic but their localization at any prime t, viewed as $\mathbf{Z}C_t$-modules, are isomorphic. Indeed, there is an overlap between the examples described in [1] and the sets of modules M_i.

We then realize the modules M_i by nilpotent spaces X_i, where $\pi_1 X_i = C$, $\pi_2 X_i = \mathbf{Z}/p^{n+k}$, and the action of π_1 on π_2 is precisely given by (0.3). Further the spaces X_i, $0 \leqq i \leqq s-1$, are all in the same genus, but no two are homotopically equivalent; moreover, we have an "Escher staircase" of regular l-sheeted coverings

$$X_0 \xrightarrow{f_0} X_1 \to \cdots \to X_i \xrightarrow{f_i} X_{i+1} \to \cdots \to X_{s-1} \xrightarrow{f_{s-1}} X_0. \tag{0.4}$$

In Section 3 we discuss the exceptional case $p = 2$, $k = 1$; and in Section 4 we modify our construction of the covering maps $f_i : X_i \to X_{i+1}$ to produce examples of finite-sheeted regular self-coverings of manifolds (which are not homotopic to a homeomorphism). We have been encouraged to examine this question by a (private) communication from Bill Goldman, in which he pointed out a connection with the study of expanding self-maps of smooth manifolds (which must therefore, by a theorem of Gromov, be infra-nilmanifolds).

We have also benefited greatly from a correspondence with Frank Adams, and conversations with Graham Higman and Craig Squier.

1. A cyclic genus

Let p be a prime, let $n, k \geqq 1$, and let $u = 1 + cp^k$, where $p \nmid c$. If we exclude the case $p = 2$, $k = 1$, we may prove

LEMMA 1.1. *The order of u modulo p^{n+k} is p^n.*

PROOF. We have

$$(1 + cp^k)^{p^n} = 1 + \sum_{r=1}^{p^n} \binom{p^n}{r} c^r p^{kr}.$$

Now if $r = ap^s$, $0 \leqq s \leqq n$, then

$$p^{n-s} \,\Big|\, \binom{p^n}{r} \qquad \left(\text{but } p^{n-s+1} \nmid \binom{p^n}{r}\right).$$

It follows that $p^{n-s+kap^s}$ is the highest power of p dividing $\binom{p^n}{r} c^r p^{kr}$; but

$$k(ap^s - 1) \geqq s, \qquad \text{since } k \geqq 1, \quad a \geqq 1, \quad p^s \geqq 1 + s,$$

so that

$$p^{n+k} \,\Big|\, \binom{p^n}{r} c^r p^{kr}$$

and hence

$$(1 + cp^k)^{p^n} \equiv 1 \bmod p^{n+k}. \tag{1.1}$$

It thus remains to show that $(1+cp^k)^{p^{n-1}} \not\equiv 1 \bmod p^{n+k}$. As above, we have

$$(1.2) \qquad (1+cp^k)^{p^{n-1}} = 1 + cp^{n+k-1} + \sum_{r=2}^{p^{n-1}} \binom{p^{n-1}}{r} c^r p^{kr},$$

and, if $r = ap^s$, $0 \leqq s \leqq n-1$, then

$$p^{n-1-s} \,\Big|\, \binom{p^{n-1}}{r} \qquad \left(\text{but } p^{n-s} \nmid \binom{p^{n-1}}{r}\right).$$

We claim that this implies that

$$p^{n+k} \,\Big|\, \binom{p^{n-1}}{r} c^r p^{kr}, \qquad r \geqq 2.$$

For we must show that $n-1-s+kap^s \geqq n+k$ or that $k(ap^s-1) \geqq s+1$. We claim that this is true, noting (i) that, if $s=0$, then $a \geqq 2$ and (ii) that we have excluded $p=2$, $k=1$, in which case the inequality would be false for $s=1$, $a=1$. It then follows that

$$(1+cp^k)^{p^{n-1}} \equiv 1 + cp^{n+k-1} \not\equiv 1 \bmod p^{n+k}$$

and the lemma is proved.

We write $u = 1 + cp^k$ and consider the group

$$N = \langle x, y; x^{p^{n+k}} = 1, yxy^{-1} = x^u \rangle.$$

PROPOSITION 1.2. *The group N is nilpotent with finite commutator subgroup.*

PROOF. Let the cyclic group C_{p^n} act on $\mathbf{Z}/p^{n+k}$ by $\xi a = ua$, where ξ generates C_{p^n}. By Lemma 1.1 this action is well-defined; it is necessarily a nilpotent action. If we let C act on $\mathbf{Z}/p^{n+k}$ via the projection $C \twoheadrightarrow C_{p^n}$, then C also acts nilpotently. Then the group N is the semidirect product of $\mathbf{Z}/p^{n+k}$ and C for this action, and hence itself nilpotent. Since $[N,N] = \langle x^{p^k} \rangle$, it is obviously finite.

REMARK. If we index the lower central series by $\Gamma_0 = N$, $\Gamma_{i+1} = [N, \Gamma_i N]$, and define the nilpotency class c to be the smallest i such that $\Gamma_i = \{1\}$, then the nilpotency class of N is the smallest integer j such that $j \geqq n/k + 1$.

Let us recall that we always exclude the case $p=2$, $k=1$. We now also exclude $p=2$, $n=1$, when the genus of N will be trivial. We may then prove

THEOREM 1.3. *The group of the genus of N is a cyclic group of order $p^{n-1}(p-1)/2$.*

PROOF. We refer to Theorem 1.4 of [2]. We first analyse the center ZN of N.

Now every element of N is expressible as $x^m y^l$; and if $[w,z] = wzw^{-1}z^{-1}$, then $[x^m y^l, x] = x^{u^l - 1}$, $[x^m y^l, y] = x^{m-um}$. Thus

$$x^m y^l \in ZN \Leftrightarrow p^{n+k} \mid u^l - 1,\ p^{n+k} \mid mp^k.$$

But, by Lemma 1.1,

$$p^{n+k} \mid u^l - 1 \Leftrightarrow p^n \mid l.$$

We conclude that $x^m y^l \in ZN \Leftrightarrow p^n \mid l,\ p^n \mid m$ so that

$$ZN = \langle x^{p^n}, y^{p^n} \rangle.$$

Then the order of the torsion subgroup of $ZN = p^k$ so that (see [2]) the *free center* of N is given by

$$FZN = \{z \in ZN;\ z = w^{p^k},\ w \in ZN\} = \langle y^{p^{n+k}} \rangle.$$

Now if $QN = N/FZN$, then the exponent of QN_{ab} is p^{n+k}. For

$$\begin{aligned} QN_{ab} &= \langle \bar{x}, \bar{y};\ p^{n+k}\bar{x} = 0, p^{n+k}\bar{y} = 0, cp^k\bar{x} = 0 \rangle \\ &= \langle \bar{x}, \bar{y};\ p^k\bar{x} = 0, p^{n+k}\bar{y} = 0 \rangle. \end{aligned}$$

We look at the semi-group of p-automorphisms of N. Let $\alpha : N \to N$ be a p-automorphism. We then have a map of exact sequences

$$\begin{array}{ccccc} C_{p^{n+k}} & \rightarrowtail & N & \twoheadrightarrow & C \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\gamma} \\ C_{p^{n+k}} & \rightarrowtail & N & \twoheadrightarrow & C \end{array}$$

and β is an automorphism, while γ is a p-automorphism. If $C = \langle \xi \rangle$, let $\gamma(\xi) = \xi^m$. Then

$$\alpha(y) = x^q y^m, \qquad \text{for some } q.$$

It follows that

$$\alpha(y^{p^{n+k}}) = x^r y^{mp^{n+k}}, \qquad \text{for some } r.$$

But since α maps FZN to FZN, we must have

$$\alpha(y^{p^{n+k}}) = y^{mp^{n+k}},$$

so that $\det \alpha = m$.

We now show that m may take precisely the values $\equiv 1 \bmod p^n$. For the constraint on m is precisely that $y^m x y^{-m} = yxy^{-1}$, i.e., that $u^m \equiv u \bmod p^{n+k}$.

But, by Lemma 1.1, this is equivalent to $m \equiv 1 \bmod p^n$. Of course, for such m, α is a p-automorphism. Thus by Theorem 1.4 of [2] the image of θ : $p-\operatorname{Aut} N \to (\mathbf{Z}/p^{n+k})^*/\{\pm 1\}$ consists of the units of $\mathbf{Z}/p^{n+k}$, mod$\{\pm 1\}$, which are $\equiv 1 \bmod p^n$. Plainly there are p^k such units so that the image of θ is a subgroup of order p^k. Now $(\mathbf{Z}/p^{n+k})^*/\{\pm 1\}$ is a cyclic group of order $p^{n+k-1}(p-1)/2$, so that the quotient group, $G(N)$, is cyclic of order $p^{n-1}(p-1)/2$.

It remains to find a generator of the group $G(N)$, given that we assign to N the role of the neutral element. Since $(\mathbf{Z}/p^n)^*/\{\pm 1\}$ is cyclic, we may find a generator l. Thus l may be regarded as a positive integer and the smallest exponent s such that $l^s \equiv \pm 1 \bmod p^n$ is $s = p^{n-1}(p-1)/2$. Here we ignore the trivial cases $p = 2$, $n = 1$; $p = 2$, $n = 2$; $p = 3$, $n = 1$, when $G(N)$ is the trivial group.

Now let $lm \equiv 1 \bmod p^n$ and let N_1 be the group given by

$$N_1 = \langle x, y\,; x^{p^{n+k}} = 1, yxy^{-1} = x^{u^m} \rangle. \tag{1.3}$$

We consider the homomorphism $\phi : N \to N_1$ given by $\phi x = x$, $\phi y = y^l$. Then ϕ gives rise, by restriction, to $\phi_F : FZN \to FZM$ with det $\phi_F = l$ and the induced map of quotient groups $QN \to QN_1$ is an isomorphism. Thus (see Proposition 1.3 in [2] or the original definition in [4]) $\delta : (\mathbf{Z}/p^{n+k})^*/\{\pm 1\} \twoheadrightarrow G(N)$ maps the class of l to N_1. Since l generates $(\mathbf{Z}/p^{n+k})^*\{\pm 1\}$ module image θ, it follows that N_1 generates the group of the genus.

Of course more is true. Let us define N_i, $0 \leqq i \leqq s-1$, where $s = p^{n-1}(p-1)/2$, by

$$N_i = \langle x, y\,; x^{p^{n+k}} = 1, yxy^{-1} = x^{u^{m^i}} \rangle. \tag{1.4}$$

Then $\phi_i : N_i \to N_{i+1}$, $0 \leqq i \leqq s-1$ $(N_s = N_0)$, given by $\phi_i x = x$, $\phi y = y^l$, embeds each N_i as a normal subgroup of N_{i+1} with quotient C_l, and, in the additive group $G(N)$, which is cyclic of order s,

$$N_i = iN_1, \qquad 0 \leqq i \leqq s-1 \quad (N_0 = N). \tag{1.5}$$

The identification (1.5) has the following remarkable consequence.

THEOREM 1.4. *Let N_i be defined by* (1.4) *and let* $(i_1, i_2, \ldots, i_t)$, $(j_1, j_2, \ldots, j_t)$ *be t-tuples of integers, in the range* $[0, s-1]$ *such that* $\Sigma_{m=1}^t i_m \equiv \Sigma_{m=1}^t j_m \bmod s$ *where* $s = p^{n-1}(p-1)/2$. *Then*

$$\prod_{m=1}^{t} N_{i_m} \cong \prod_{m=1}^{t} N_{j_m}.$$

PROOF. This is an immediate application of Theorem 3.2 of [2].

We now involve another result from [2]. We know from Corollary 2.2 of that paper that there exists, for any i, k, an l-equivalence $\psi : N_i \to N_k$. Let $\phi : N_j \to N_k$ be given by $\phi x = x$, $\phi y = y^{l^{k-j}}$, where $k - j$ is computed modulo s. Then Theorem 2.8 of [2] implies

THEOREM 1.5. *For any l-equivalence $\psi : N_i \to N_k$, the pull-back of $\psi : N_i \to N_k$ and $\phi : N_j \to N_k$ is N_t, where $k + t \equiv i + j \bmod s$.*

Similarly, we have

THEOREM 1.6. *For any l-equivalence $\psi : N_j \to N_i$, the push-out of $\psi : N_j \to N_i$ and $\phi : N_j \to N_k$ is N_t, where $j + t \equiv i + k \bmod s$.*

2. Realizing the genus of N

Let the group $C = \langle \xi \rangle$ act on the abelian group $\mathbf{Z}/p^{n+k}$ by

$$\xi a = u^{m^i} a, \qquad 0 \leqq i \leqq s-1 \tag{2.1}$$

where the integers m, s have the same meaning as in Section 1. Then the semidirect product of $\mathbf{Z}/p^{n+k}$ and C, for this action, is precisely the group N_i. Let us write A_i for the C-module described above. Then plainly the modules $A_0, A_1, \dots, A_{s-1}$ are pairwise non-isomorphic, but all are in the same genus (compare [1]).

Our objective in this section is to realize the modules A_i as homotopy groups of nilpotent polyhedra. Thus we will construct nilpotent polyhedra $X_0, X_1, \dots, X_{s-1}$ and l-sheeted regular covering maps

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \to X_{s-1} \xrightarrow{f_{s-1}} X_0 \tag{2.2}$$

such that (i) $\pi_1 X_i = C = \langle \xi \rangle$; (ii) $\pi_2 X_i = \mathbf{Z}/p^{n+k}$; (iii) $\pi_1 X_i$ acts on $\pi_2 X_i$ by (2.1); (iv) f_i induces an injection of $\pi_1 X_i$ in $\pi_1 X_{i+1}$ with quotient cyclic of order l; (v) all X_i are in the same genus; (vi) no two of $X_0, X_1, \dots, X_{s-1}$ are homotopically equivalent.

We begin with a construction of greater generality and then specialize to achieve our objective. Let M be a connected polyhedron with $\pi_1 M$ cyclic of order k_0. Then $H^2(M; \mathbf{Z})$ contains the summand $\operatorname{Ext}(\mathbf{Z}/k_0, \mathbf{Z}) = \mathbf{Z}/k_0$. Let g be a generator of this group. We may represent g by a map, which we also designate g, from M to $K(\mathbf{Z}, 2)$. We use g to induce a principal circle-bundle X over M. Thus we have the sequence of maps

$$S^1 \longrightarrow X \xrightarrow{h} M \xrightarrow{g} K(\mathbf{Z}, 2). \tag{2.3}$$

Now (2.3) induces, in 1-dimensional homology, the short exact sequence

$$\mathbf{Z} \rightarrowtail H_1X \twoheadrightarrow \mathbf{Z}/k_0. \tag{2.4}$$

This extension represents the element $g \in \mathrm{Ext}(\mathbf{Z}/k_0, \mathbf{Z})$. If we apply π_1 to (2.3) we obtain the central extension†

$$C \rightarrowtail \pi_1 X \twoheadrightarrow C_{k_0}. \tag{2.5}$$

Then π_1X must be abelian, so that (2.4) and (2.5) effectively coincide. Moreover, since g generates $\mathrm{Ext}(\mathbf{Z}/k_0, \mathbf{Z})$, we know that $H_1X = \mathbf{Z}$, whence $\pi_1X = C$; and the embedding $C \rightarrowtail \pi_1X$ of (2.5) maps the generator to the k_0th power of the generator.

Now let l be prime to k_0; we may regard l as a map $K(\mathbf{Z},2) \to K(\mathbf{Z},2)$ and thus obtain the diagram

$$\begin{array}{ccccccc} S^1 & \longrightarrow & X & \xrightarrow{h} & M & \xrightarrow{g} & K(\mathbf{Z},2) \\ \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle f} & & \| & & \downarrow{\scriptstyle l} \\ S^1 & \longrightarrow & X_1 & \xrightarrow{h_1} & M & \xrightarrow{lg} & K(\mathbf{Z},2) \end{array} \tag{2.6}$$

We establish the following properties of diagram (2.6).

PROPOSITION 2.1. *The map $f: X \to X_1$ is (homotopically) a regular covering map; indeed $f_*\pi_1X$ is normal in π_1X_1 with quotient C_l.*

PROOF. Since g represents an element of $\mathrm{Ext}(H_1M, \mathbf{Z})$ it follows that $g_* = 0: H_2M \to \mathbf{Z}$, so that $g_* = 0: \pi_2M \to \mathbf{Z}$. It thus follows easily that $h_*: \pi_iX \cong \pi_iM$, $i \geqq 2$; likewise $h_{1*}: \pi_iX_1 \cong \pi_iM$, $i \geqq 2$, so that $f_*: \pi_iX \cong \pi_iX_1$, $i \geqq 2$. Moreover, applying π_1 to (2.6), we get the map of central extensions

$$\begin{array}{ccccc} C & \xrightarrow{k_0} & C & \twoheadrightarrow & C_{k_0} \\ \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle f_*} & & \| \\ C & \rightarrowtail & \pi_1X_1 & \twoheadrightarrow & C_{k_0} \end{array} \tag{2.7}$$

Since l is prime to k_0, it is plain that $\pi_1X_1 = C$ and we may write $f_* = l$. This completes the proof.

PROPOSITION 2.2. *If M is nilpotent, so are X, X_1; and X, X_1 are in the same genus.*

† We use multiplicative notation (and, therefore, $C, C_{k_0}, \ldots$) for the fundamental group, even where it is commutative.

PROOF. Since $\pi_1 X$ acts on $\pi_i X$, $i \geqq 2$, through the action of $\pi_1 M$ on $\pi_i M$, it follows that the action is nilpotent. Since, further, $\pi_1 X$ is commutative, it follows that X is nilpotent. Similarly, X_1 is nilpotent.

Let q be a prime. Then q is prime to k_0 or to l. If q is prime to k_0, then $g_q \simeq 0$, so that $X_q \simeq (M \times S^1)_q$; and, likewise, $X_{1q} \simeq (M \times S^1)_q$. If q is prime to l, then l_q is invertible, so that $f_q : X_q \simeq X_{1q}$. Thus X, X_1 are in the same genus.

We claim that X, X_1 are not, in general, of the same homotopy type. To establish this, we specialize our construction. Let $k_0 = p^n$, $n \geqq 1$. We assume that $\pi_2 M = \mathbf{Z}/p^{n+k}$, $k \geqq 1$, where we exclude the case $p = 2$, $k = 1$, and that $\pi_1 M = \langle \eta \rangle$ acts on $\pi_2 M$ by $\eta a = ua$, where $u = 1 + cp^k$, $p \nmid c$. This is valid by Lemma 1.1. Choose a generator ξ of $\pi_1 X$ mapping onto η. Then $\pi_1 X$ acts on $\pi_2 X$ by $\xi a = ua$. Now choose l prime to p and let m be such that $lm \equiv 1 \bmod p^n$. We construct diagram (2.6) and it follows that if $f_* \xi = \xi_1^l$, where ξ_1 generates $\pi_1 X_1$, then $\xi_1^l a = ua$, so that

$$\xi a = ua, \qquad \xi_1 a = u^m a. \tag{2.8}$$

It follows that X, X_1 will have different homotopy types unless either $p^{n+k} \mid u^m - u$ or $p^{n+k} \mid u^{-m} - u$. Lemma 1.1 ensures that this can only happen if $m \equiv \pm 1 \bmod p^n$ or, equivalently, $l \equiv \pm 1 \bmod p^n$. Thus we infer

THEOREM 2.3. *If $\pi_1 M = C_{p^n}$, $\pi_2 M = \mathbf{Z}/p^{n+k}$, where we exclude the case $p = 2$, $k = 1$, and if $\pi_1 M = \langle \eta \rangle$ acts on $\pi_2 M$ by $\eta a = ua$, where $u = 1 + cp^k$, $p \nmid c$, then in* (2.6), *X and X_1 have different homotopy types provided that $l \not\equiv \pm 1 \bmod p^n$.*

Notice that such an l may be found, provided that we further exclude $p = 2$, $n = 1$; $p = 2$, $n = 2$; $p = 3$, $n = 1$. Indeed, of course, we may go further. If we choose l, as in Section 1, to represent a generator of $(\mathbf{Z}/p^n)^*/\{\pm 1\}$, then $f : X \to X_1$ in (2.6) is precisely the map $f_0 : X_0 \to X_1$ we seek in (2.2); and we obtain the entire cycle (2.2) by repeatedly using $l : K(\mathbf{Z}, 2) \to K(\mathbf{Z}, 2)$ as in (2.6). Thus the "general term" is

$$\begin{array}{ccccccc} S^1 & \longrightarrow & X_i & \xrightarrow{h_i} & M & \xrightarrow{l^i g} & K(\mathbf{Z}, 2) \\ \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle f_i} & & \| & & \downarrow{\scriptstyle l} \\ S^1 & \longrightarrow & X_{i+1} & \xrightarrow{h_{i+1}} & M & \xrightarrow{l^{i+1} g} & K(\mathbf{Z}, 2) \end{array} \tag{2.9}$$

where $\pi_1 M = C_{p^n}$, acting on $\pi_2 M = \mathbf{Z}/p^{n+k}$ by $\eta a = ua$, $u = 1 + cp^k$, $p \nmid c$. Then the C-module $\pi_2 X_i$ is just A_i, and the semidirect product of $\mathbf{Z}/p^{n+k}$ and C for this action is precisely the group N_i of Section 1.

3. The exceptional case

Lemma 1.1 fails for the case $p = 2$, $k = 1$. For example, if $u = 7$ then the order of $u \bmod 2^3$ is 2, not 2^2. To state a corresponding lemma for this exceptional case, let us consider positive integers u of the form $1 + 2c$, with c odd — in other words, integers which are congruent to $3 \bmod 4$. For such an integer u there is a unique m such that

$$u \equiv 2^m - 1 \bmod 2^{m+1}, \tag{3.1}$$

and $m \geqq 2$. (If we write u in base 2, then m is the number of 1's we get, starting on the right, before meeting the first 0.)

We may then prove

LEMMA 3.1. *The order of u modulo 2^{n+1} is 2^{n+1-m}, provided $n \geqq m$.*

PROOF. Throughout this argument, v will stand for an arbitrary variable integer. Thus, (3.1) says that

$$u = 2^m - 1 + v2^{m+1}.$$

Squaring, we get

$$u^2 = 1 - 2^{m+1} + v2^{m+2}, \qquad \text{since } m \geqq 2.$$

It is now easy to prove, inductively, that

$$u^{2^r} = 1 - 2^{m+r} + v2^{m+r+1}, \qquad \text{provided } r \geqq 1. \tag{3.2}$$

Then (3.2) implies that the order of $u \bmod 2^{m+r}$ is 2^r, provided[†] $r \geqq 1$, which is equivalent to the statement of the lemma.

We may view this result in a slightly different light. Let $u = 1 + c2^k$, c odd, $k \geqq 2$. Then Lemma 1.1 tells us that the order of u modulo 2^{n+k} is 2^n. Lemma 3.1 implies that, though we must exclude $p = 2$, $k = 1$ from consideration, we may, by way of compensation, allow $u = -1 + c2^k$, c odd, $k \geqq 2$, when $p = 2$, and still conclude that the order of u modulo 2^{n+k} is 2^n.

4. Self-covering manifolds

The general construction of Section 2 plainly permits us to describe a family of polyhedra X admitting non-trivial self-covering maps $f : X \to X$. (Of course, our

[†] The case $r = 1$ again requires the observation that $m \geqq 2$.

object in Section 2 was very different, namely, to obtain coverings $f: X \to X_1$ where X_1 was not homotopy-equivalent to X.) Recall that we chose a positive integer k_0 and a connected polyhedron M with $\pi_1 M = C_{k_0}$; and we then constructed a principal S^1-bundle

$$S^1 \longrightarrow X \xrightarrow{h} M \xrightarrow{g} K(\mathbf{Z}, 2).$$

Let us now choose l so that $l \equiv \pm 1 \bmod k_0$, to obtain the diagram (as a special case of (2.6))

$$\begin{array}{ccccccc} S^1 & \longrightarrow & X & \xrightarrow{h} & M & \xrightarrow{g} & K(\mathbf{Z},2) \\ \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle f} & & \| & & \downarrow{\scriptstyle l} \\ S^1 & \longrightarrow & X_1 & \xrightarrow{h_1} & M & \xrightarrow{lg} & K(\mathbf{Z},2) \end{array} \tag{4.1}$$

We now observe that $X \simeq X_1$. If $l \equiv 1 \bmod k_0$, this is obvious, since then $lg = g$. But if $l \equiv -1 \bmod k_0$, it is also true that $X \simeq X_1$, since if $l = -1$ in (3.1), then l is a homotopy equivalence and so therefore is f. We have proved

THEOREM 4.1. *If* $l \equiv \pm 1 \bmod k_0$, *then the map* f *of* (4.1) *is* (*homotopically*) *a regular* l*-sheeted covering map.*

Note that we may take X to be a manifold, provided M is a manifold. Thus an example of this construction is obtained by taking $M = \mathbf{R}P^2$, so that $k_0 = 2$. The manifold X will then be a circle bundle over $\mathbf{R}P^2$. Then, for any odd positive integer l, we obtain a regular l-sheeted covering map $f: X \to X$.

Now the construction given in Theorem 4.1 is based on the fact that the *fiber* S^1 admits "expanding" maps. We close by giving an example where the space X admitting such non-trivial self-coverings is regarded as a fiber space over the base S^1 and we exploit the expanding maps of the *base.*

Our example is the *Klein bottle.* This is, apart from the trivial example of the torus, the only possible closed surface that can admit finite-sheeted self-coverings; for it is plain that if a compact polyhedron admits such self-coverings its Euler characteristic must be zero.

Since the Klein bottle K is an Eilenberg–MacLane space $K(G, 1)$, it is obvious that K admits self-coverings if and only if G has subgroups isomorphic to itself. Now the group G (the fundamental group of K) admits the presentation

$$G = \langle x, y\,;\, yxy^{-1} = x^{-1} \rangle.$$

If we change to the generators a, b where $a = xy$, $b = y$, we obtain the presentation

$$G = \langle a, b\, ; a^2 = b^2 \rangle. \tag{4.2}$$

The following theorem will imply that G admits subgroups — in fact, normal subgroups — isomorphic to itself.

THEOREM 4.2. *Let $G = \langle a, b\, ; a^k = b^k \rangle$, let l be prime to k, and let $H = \langle a^l, b^l \rangle$. Then* (i) *$H \lhd G$ with $G/H = C_l$*; (ii) *$H \cong G$.*

PROOF. (i) It suffices, first, to show that $b^{-1}a^l b \in H$. For then, by symmetry, $a^{-1}b^l a \in H$ and so $H \lhd G$. Now $\exists$ integers u, v with $uk + vl = 1$. Then, noting that a^k $(= b^k)$ is in the center of G,

$$b^{-1}a^l b = b^{-uk}b^{-vl}a^l b^{vl} b^{uk} = b^{-vl}a^l b^{vl} \in H.$$

Now $G/H = \langle \bar{a}, \bar{b}\, ; \bar{a}^k = \bar{b}^k, \bar{a}^l = 1, \bar{b}^l = 1 \rangle$. Since $\bar{a}^k = \bar{b}^k, \bar{a}^l = \bar{b}^l$, it follows that $\bar{a} = \bar{b}$, so that $G/H = \langle \bar{a}\, ; \bar{a}^l = 1 \rangle$.

(ii) Write $c = a^k = b^k$. Then the elements of G have the normal form

$$c^q a^{r_1} b^{s_1} \cdots a^{r_n} b^{s_n}, \quad 0 \leqq r_i,\ s_j \leqq k-1, \text{ only } r_1, s_n \text{ may be zero.} \tag{4.3}$$

Write $A = a^l$, $B = b^l$, $C = c^l$. Then certainly every element of H may be written in the form

$$C^q A^{r_1} B^{s_1} \cdots A^{r_n} B^{s_n}, \quad 0 \leqq r_i,\ s_j \leqq k-1, \text{ only } r_1, s_n \text{ may be zero,} \tag{4.4}$$

and the theorem is proved if we can show that such an expression (4.4) is unique.

Now suppose $0 \leqq r \leqq k-1$ and let $rl = mk + r'$, $0 \leqq r' \leqq k-1$; likewise, suppose $0 \leqq \bar{r} \leqq k-1$ and let $\bar{r}l = \bar{m}k + \bar{r}'$, $0 \leqq \bar{r}' \leqq k-1$. We claim that $r = \bar{r} \Leftrightarrow r' = \bar{r}'$. Obviously $r = \bar{r} \Rightarrow r' = \bar{r}'$; but, conversely, if $r' = \bar{r}'$, then $(r - \bar{r})l = (m - \bar{m})k$, so that $k \mid (r - \bar{r})l$, $k \mid (r - \bar{r})$, $r = \bar{r}$. From this observation it immediately follows from the uniqueness of (4.3) that if

$$C^q A^{r_1} B^{s_1} \cdots A^{r_n} B^{s_n} = C^{\bar{q}} A^{\bar{r}_1} B^{\bar{s}_1} \cdots A^{\bar{r}_{\bar{n}}} B^{\bar{s}_{\bar{n}}},$$

both sets of exponents subject to the canonical restrictions, then $n = \bar{n}$, $r_i = \bar{r}_i$, $s_j = \bar{s}_j$. Thus $C^q = C^{\bar{q}}$, so that $q = \bar{q}$, and the theorem is proved.

That we cannot expect, in this way, to get examples (as in Section 1) of mutually covering, non-isomorphic groups, is suggested by the following Theorem.

THEOREM 4.3. *Let $G = \langle a, b\, ; a^k = b^k \rangle$, let m, n be prime to k, and let $\bar{H} = \langle a^m, b^n \rangle$. Then $\bar{H} = H = \langle a^l, b^l \rangle$, where $l = \gcd(m, n)$.*

PROOF. We first show that $\bar{H} \lhd G$. As in Theorem 4.2 it suffices to show that $b^{-1}a^m b \in \bar{H}$, and this follows because n is prime to k. Now $\bar{H} \subseteq H$, so we have the commutative diagram

$$\begin{array}{ccccc} \bar{H} & \rightarrowtail & G & \twoheadrightarrow & G/\bar{H} \\ \downarrow & & \| & & \downarrow \\ H & \rightarrowtail & G & \twoheadrightarrow & G/H \end{array}$$

and it suffices to show that the cardinality of $G/\bar{H}$ satisfies

$$|G/\bar{H}| \leqq l. \tag{4.5}$$

Now $G/\bar{H} = \langle \bar{a}, \bar{b}; \bar{a}^k = \bar{b}^k, \bar{a}^m = 1, \bar{b}^n = 1 \rangle$. If $uk + vn = 1$, then $\bar{b} = \bar{b}^{uk} = \bar{a}^{uk}$, so that $G/\bar{H}$ is cyclic, generated by $\bar{a}$, and the order of $G/\bar{H}$ is thus a divisor of m. Similarly the order of $G/\bar{H}$ is a divisor of n, so that it is a divisor of l, and (4.5) is proved.

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